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Efficient Higher Order Composite Plate Theory for General Lamination Configurations

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An efficient higher order plate theory for laminated composites is developed. A composite plate theory for general lamination configurations is obtained by superimposing a cubic varying displacement field on a zig-zag linearly varying displacement. The theory has the same number of dependent unknowns as first-order shear deformation theory, and the number of unknowns is independent of the number of layers. The displacement satisfies transverse shear stress continuity conditions at the interface between layers as well as shear free surface conditions. Thus, an artificial shear correction factor is not needed. To demonstrate and compare with other theories, the analytical solution for cylindrical bending is obtained. The present theory gives deflections and stresses that compare well with other known theories.

Introduction

As the advancement in technologies of manufacturing and materials continues, the engineer encounters new challenges. In recent years advances in technology allowed the transition of composite materials from secondary to primary structural components. Consequently, the current application of composite materials demands development of thicker composite structures to sustain heavier loads. The anisotropy and inhomogeneity of the materials require improved analysis of thick laminated composite structures. Since these structures are often modeled as plates or shells, much attention has focused on plate/shell models that adequately describe thick laminated behavior.

When classical plate theory is used to analyze laminated composites, deflections are underestimated and natural frequencies and buckling loads are overestimated. This result is a consequence of neglecting transverse shear deformations, which introduces a constraint that stiffens the structure. To improve the situation, various refined plate theories have been developed. Among these, polynomial based higher order theories[1-3] and first-order zig-zag theories[4,5] are quite successful in obtaining global behavior. However, this success is limited to symmetric laminations. In the case of asymmetry, results are less satisfactory due to physically unrealistically assumed displacements. Therefore these theories are inadequate for the analysis of general lamination configurations.

On the other hand, discretized plate theories,[6,8,9] in both symmetric and asymmetric layups, successfully predict both global behavior and the stress distribution through the thickness. However, the number of unknowns depends on the number of layers, and so these theories are inefficient for analyzing thick plates with many layers.

The motivation for the present work is the need for a plate theory, with a minimal number of dependent variables, that can accurately predict both global and through the thickness behavior in thick multilayered plates without lamination symmetry. As in classical theory we use the midplane as our reference surface, and we assume a combination of a linear zig-zag and a cubic varying in-plane displacement field. The cubic variation accounts for the overall parabolic transverse shear distribution known from single layer theory, and the zig-zag accounts for the strain discontinuities that are required between layers to satisfy stress continuity conditions. The linear zig-zag has a different slope in each layer. Transverse normal strains are neglected.

Formulation

The assumed displacement field can be expressed in the form

$$u_\alpha = u_\alpha^o + \sum_{k=0}^{n_u-1} S_\alpha^k(z-z_k)H(z-z_k) + \sum_{k=0}^{n_l-1} T_\alpha^k(z-\zeta_k)H(-z+\zeta_k) + \xi_\alpha z^2 + \phi_\alpha z^3$$

$$u_3 = w(x_1,x_2) \tag{1}$$

where u_α^o and w denote the displacements of a point (x_1, x_2) on the midplane, n_u is the number of layers in the upper half, n_l is the number of layers in the lower half, and $H(z-z_k)$ is the Heaviside unit step function. The lamination layup and displacement configurations are shown in Fig. 1. We define ψ_α as the rotation relative to the midplane of a fiber initially normal to the midplane, evaluated on the z^+ side at the midplane.

$$\frac{\partial u_\alpha}{\partial z}\Big|_{z=0^+} = S_\alpha^o = \psi_\alpha \tag{2}$$

We now impose shear stress free boundary conditions for the upper and lower surface of the plate: $\sigma_{3\alpha}|_{z=\pm h/2} = 0$. For orthotropic layers, the shear stresses $\sigma_{3\alpha}$ depend only on the transverse shear strains, so the traction free condition can be satisfied by

$$\phi_\alpha = -\frac{4}{3h^2}\left\{w_{,\alpha} + \frac{1}{2}\left(\sum_{k=0}^{n_u-1} S_\alpha^k + \sum_{k=0}^{n_l-1} T_\alpha^k\right)\right\}$$

$$\xi_\alpha = -\frac{1}{2h}\left(\sum_{k=0}^{n_u-1} S_\alpha^k - \sum_{k=0}^{n_l-1} T_\alpha^k\right) \tag{3}$$

Then imposing transverse shear continuity conditions between layers leads to linear algebraic equations that can be solved for the change in slope S_α^k and T_α^k between layers in terms of the angle change ψ_α and midplane slope $w_{,\alpha}$.

$$S_\alpha^k = a_{\alpha\gamma}^k(w_{,\gamma} + \psi_\gamma) + b_{\alpha\gamma}^k w_{,\gamma}$$

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where

$$k=0,\ 1,\ 2,\ \ldots,\ n_u-1$$

$$T_\alpha^k = c_{\alpha\gamma}^k(w_{,\gamma}+\psi_\gamma)+d_{\alpha\gamma}^k w_{,\gamma} \tag{4}$$

where $k=0,\ 1,\ 2,\ \ldots,\ n_l-1$ and $a_{\alpha\gamma}^k$, $b_{\alpha\gamma}^k$, $c_{\alpha\gamma}^k$, and $d_{\alpha\gamma}^k$ are functions of material properties and laminate thicknesses. The calculation of $a_{\alpha\gamma}^k$, $b_{\alpha\gamma}^k$, $c_{\alpha\gamma}^k$, and $d_{\alpha\gamma}^k$ is described in the Appendix.

The displacement field in Eq. (1) becomes

$$\begin{aligned}
u_\alpha = u_\alpha^o &+ \sum_{k=0}^{n_u-1}[a_{\alpha\gamma}^k(w_{,\gamma}+\psi_\gamma)+b_{\alpha\gamma}^k w_{,\gamma}](z-z_k)H(z-z_k) \\
&+ \sum_{k=0}^{n_l-1}[c_{\alpha\gamma}^k(w_{,\gamma}+\psi_\gamma)+d_{\alpha\gamma}^k w_{,\gamma}](z-\varsigma_k)H(-z+\varsigma_k) \\
&- \frac{z^2}{2h}\left\{\sum_{k=0}^{n_u-1}[a_{\alpha\gamma}^k(w_{,\gamma}+\psi_\gamma)+b_{\alpha\gamma}^k w_{,\gamma}]\right. \\
&\left.- \sum_{k=0}^{n_l-1}[c_{\alpha\gamma}^k(w_{,\gamma}+\psi_\gamma)+d_{\alpha\gamma}^k w_{,\gamma}]\right\} \\
&- \frac{4z^3}{3h^2}\left(w_{,\alpha}+\frac{1}{2}\left\{\sum_{k=0}^{n_u-1}[a_{\alpha\gamma}^k(w_{,\gamma}+\psi_\gamma)+b_{\alpha\gamma}^k w_{,\gamma}]\right.\right. \\
&\left.\left.+ \sum_{k=0}^{n_l-1}[c_{\alpha\gamma}^k(w_{,\gamma}+\psi_\gamma)+d_{\alpha\gamma}^k w_{,\gamma}]\right\}\right)
\end{aligned}$$

$$u_3 = w(x_1,\ x_2) \tag{5}$$

Thus, the final form of the displacement fields are expressed in terms of the midsurface in-plane stretchings, rotations, and deflection. The constitutive equations for an orthotropic layer, in the principal axes system of the material, are

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & 0\\ \bar{Q}_{12} & \bar{Q}_{22} & 0\\ 0 & 0 & \bar{Q}_{66}\end{bmatrix}\begin{Bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \gamma_{12}\end{Bmatrix} \tag{6}$$

$$\begin{Bmatrix}\sigma_{32}\\ \sigma_{31}\end{Bmatrix} = \begin{bmatrix}\bar{Q}_{44} & 0\\ 0 & \bar{Q}_{55}\end{bmatrix}\begin{Bmatrix}\gamma_{32}\\ \gamma_{31}\end{Bmatrix} \tag{7}$$

where $\bar{Q}_{ij}$ are the plane stress reduced elastic moduli in the material axes of the layer. Since the x, y coordinate system will not, in general, be aligned with the material axes of each layer,

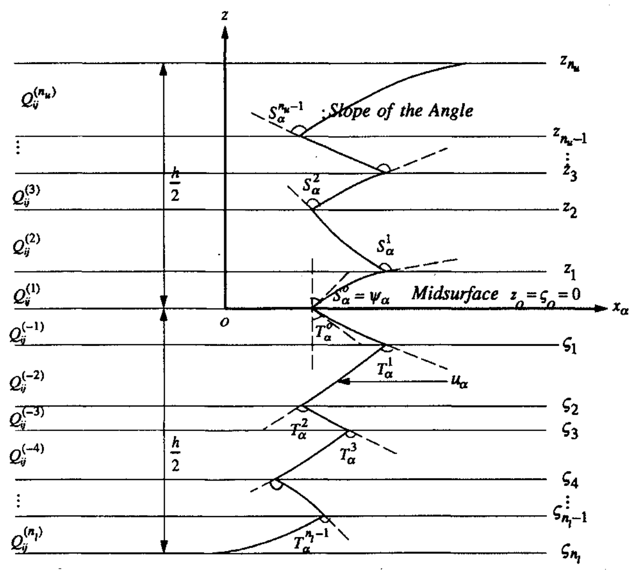

Fig. 1 General lamination layup and displacement configuration.

these constitutive equations must be transformed to the x, y system in each lamina. Thus, in each lamina, the constitutive equations become

$$\begin{Bmatrix}\sigma_{11}^{(k)}\\ \sigma_{22}^{(k)}\\ \sigma_{12}^{(k)}\end{Bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & Q_{16}\\ Q_{12} & Q_{22} & Q_{26}\\ Q_{16} & Q_{26} & Q_{66}\end{bmatrix}^{(k)}\begin{Bmatrix}\epsilon_{11}^{(k)}\\ \epsilon_{22}^{(k)}\\ \gamma_{12}^{(k)}\end{Bmatrix} \tag{8}$$

$$\begin{Bmatrix}\sigma_{32}^{(k)}\\ \sigma_{31}^{(k)}\end{Bmatrix} = \begin{bmatrix}Q_{44} & Q_{45}\\ Q_{45} & Q_{55}\end{bmatrix}^{(k)}\begin{Bmatrix}\gamma_{32}^{(k)}\\ \gamma_{31}^{(k)}\end{Bmatrix} \tag{9}$$

where Q_{ij} are the transformed material moduli for each lamina.

Using the virtual work principle, equilibrium equations and consistent boundary conditions can be obtained.

$$\begin{aligned}
&\int_v \sigma_{ij}\,\delta\epsilon_{ij}\,\mathrm{d}v - \int_A p\,\delta w\,\mathrm{d}A \\
&= \int_A N_{\alpha\beta}\,\delta u_{\alpha,\beta}^o + \sum_{k=0}^{n_u-1} a_{\alpha\gamma}^k(M_{\alpha\beta}^{+k}-z_k N_{\alpha\beta}^{+k})(\delta w_{,\gamma\beta}+\delta\psi_{\gamma,\beta}) \\
&+ \sum_{k=0}^{n_u-1} b_{\alpha\gamma}^k(M_{\alpha\beta}^{+k}-z_k N_{\alpha\beta}^{+k})\delta w_{,\gamma\beta} \\
&+ \sum_{k=0}^{n_l-1} c_{\alpha\gamma}^k(M_{\alpha\beta}^{-k}-\varsigma_k N_{\alpha\beta}^{-k})(\delta w_{,\gamma\beta}+\delta\psi_{\gamma,\beta}) + \sum_{k=0}^{n_l-1} d_{\alpha\gamma}^k(M_{\alpha\beta}^{-k} \\
&- \varsigma_k N_{\alpha\beta}^{-k})\delta w_{,\gamma\beta} - \frac{2}{3h^2}R_{\alpha\beta}^{(3)}\left\{\sum_{k=0}^{n_u-1}[a_{\alpha\gamma}^k(\delta w_{,\gamma\beta}+\delta\psi_{\gamma,\beta})\right. \\
&\left.+ b_{\alpha\gamma}^k\delta w_{,\gamma\beta}] + \sum_{k=0}^{n_l-1}[c_{\alpha\gamma}^k(\delta w_{,\gamma\beta}+\delta\psi_{\gamma,\beta}) + d_{\alpha\gamma}^k\delta w_{,\gamma\beta}] + 2\delta w_{,\gamma\beta}\right\} \\
&- \frac{1}{2h}R_{\alpha\beta}^{(2)}\left\{\sum_{k=0}^{n_u-1}[a_{\alpha\gamma}^k(\delta w_{,\gamma\beta}+\delta\psi_{\gamma,\beta}) + b_{\alpha\gamma}^k\delta w_{,\gamma\beta}]\right. \\
&\left.- \sum_{k=0}^{n_l-1}[c_{\alpha\gamma}^k(\delta w_{,\gamma\beta}+\delta\psi_{\gamma,\beta}) + d_{\alpha\gamma}^k\delta w_{,\gamma\beta}]\right\} + Q_\alpha\delta w_{,\alpha} \\
&+ \sum_{k=0}^{n_u-1} a_{\alpha\gamma}^k Q_\alpha^{+k}(\delta w_{,\gamma}+\delta\psi_\gamma) + \sum_{k=0}^{n_u-1} b_{\alpha\gamma}^k Q_\alpha^{+k}\delta w_{,\gamma} \\
&+ \sum_{k=0}^{n_l-1} c_{\alpha\gamma}^k Q_\alpha^{-k}(\delta w_{,\gamma}+\delta\psi_\gamma) + \sum_{k=0}^{n_l-1} d_{\alpha\gamma}^k Q_\alpha^{-k}\delta w_{,\gamma} \\
&- \frac{1}{h}V_\alpha^{(1)}\left\{\sum_{k=0}^{n_u-1}[a_{\alpha\gamma}^k(\delta w_{,\gamma}+\delta\psi_\gamma) + b_{\alpha\gamma}^k\delta w_{,\gamma}]\right. \\
&\left.- \sum_{k=0}^{n_l-1}[c_{\alpha\gamma}^k(\delta w_{,\gamma}+\delta\psi_\gamma) + d_{\alpha\gamma}^k\delta w_{,\gamma}]\right\} \\
&- \frac{4}{h^2}V_\alpha^{(2)}\left\{\delta w_{,\alpha} + \frac{1}{2}\sum_{k=0}^{n_u-1}[a_{\alpha\gamma}^k(\delta w_{,\gamma}+\delta\psi_\gamma) + b_{\alpha\gamma}^k\delta w_{,\gamma}]\right. \\
&\left.+ \frac{1}{2}\sum_{k=0}^{n_l-1}[c_{\alpha\gamma}^k(\delta w_{,\gamma}+\delta\psi_\gamma) + d_{\alpha\gamma}^k\delta w_{,\gamma}]\right\} - p\,\delta w\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= 0
\end{aligned} \tag{10}$$

Table 1 Center deflection $\bar{w}$ for antisymmetric four-layer cross-ply laminates (0/90/0/90)

L/h	Elasticity	Present Theory	First-order zig-zag	LCW[1]
4	4.181	4.083	3.316	3.587
6	2.562	2.501	2.107	2.242

where stress resultants are defined as follows

$$(N_{\alpha\beta}, M_{\alpha\beta}, R^{(2)}_{\alpha\beta}, R^{(3)}_{\alpha\beta}) = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}(1, z, z^2, z^3)\, dz$$

$$(Q_\alpha, V^{(1)}_\alpha, V^{(2)}_\alpha) = \int_{-h/2}^{h/2} \sigma_{3\alpha}(1, z, z^2)\, dz$$

$$(N^{+k}_{\alpha\beta}, M^{+k}_{\alpha\beta}) = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}(1, z)\, H(z-z_k)\, dz = \int_{z_k}^{h/2} \sigma_{\alpha\beta}(1, z)\, dz$$

$$(N^{-k}_{\alpha\beta}, M^{-k}_{\alpha\beta}) = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}(1, z)\, H(-z+\zeta_k)\, dz = \int_{-h/2}^{\zeta_k} \sigma_{\alpha\beta}(1, z)\, dz$$

$$Q^{+k}_\alpha = \int_{-h/2}^{h/2} \sigma_{3\alpha}\, H(z-z_k)\, dz = \int_{z_k}^{h/2} \sigma_{3\alpha}\, dz$$

$$Q^{-k}_\alpha = \int_{-h/2}^{h/2} \sigma_{3\alpha}\, H(-z+\zeta_k)\, dz = \int_{-h/2}^{\zeta_k} \sigma_{3\alpha}\, dz$$

Integrating by parts in Eq. (10), we obtain the equilibrium equations

$$\delta u^o_\alpha \;:\; N_{\alpha\beta,\beta} = 0 \tag{11}$$

$$\begin{aligned}
\delta\psi_\alpha \;:\; & -\sum_{k=0}^{n_u-1} a^k_{\gamma\alpha}(M^{+k}_{\gamma\beta,\beta} - z_k N^{+k}_{\gamma\beta,\beta}) - \sum_{k=0}^{n_l-1} c^k_{\gamma\alpha}(M^{-k}_{\gamma\beta,\beta} - \zeta_k N^{-k}_{\gamma\beta,\beta}) \\
& + \frac{2}{3h^2} R^{(3)}_{\gamma\beta,\beta}\left(\sum_{k=0}^{n_u-1} a^k_{\gamma\alpha} + \sum_{k=0}^{n_l-1} c^k_{\gamma\alpha}\right) \\
& + \frac{1}{2h} R^{(2)}_{\gamma\beta,\beta}\left(\sum_{k=0}^{n_u-1} a^k_{\gamma\alpha} + \sum_{k=0}^{n_l-1} c^k_{\gamma\alpha}\right) + \sum_{k=0}^{n_u-1} a^k_{\gamma\alpha} Q^{+k}_\gamma \\
& + \sum_{k=0}^{n_l-1} c^k_{\gamma\alpha} Q^{-k}_\gamma - \frac{1}{h} V^{(1)}_\gamma \left(\sum_{k=0}^{n_u-1} a^k_{\gamma\alpha} - \sum_{k=0}^{n_l-1} c^k_{\gamma\alpha}\right) \\
& - \frac{4}{h^2} V^{(2)}_\gamma \left(\frac{1}{2}\sum_{k=0}^{n_u-1} a^k_{\gamma\alpha} + \frac{1}{2}\sum_{k=0}^{n_l-1} c^k_{\gamma\alpha}\right) = 0
\end{aligned} \tag{12}$$

$$\begin{aligned}
\delta w \;:\; & \sum_{k=0}^{n_u-1} a^k_{\alpha\gamma}(M^{+k}_{\alpha\beta,\gamma\beta} - z_k N^{+k}_{\alpha\beta,\gamma\beta}) + \sum_{k=0}^{n_u-1} b^k_{\alpha\gamma}(M^{+k}_{\alpha\beta,\gamma\beta} - z_k N^{+k}_{\alpha\beta,\gamma\beta}) \\
& + \sum_{k=0}^{n_l-1} c^k_{\alpha\gamma}(M^{-k}_{\alpha\beta,\gamma\beta} - \zeta_k N^{-k}_{\alpha\beta,\gamma\beta}) + \sum_{k=0}^{n_l-1} d^k_{\alpha\gamma}(M^{-k}_{\alpha\beta,\gamma\beta} - \zeta_k N^{-k}_{\alpha\beta,\gamma\beta}) \\
& - \frac{2}{3h^2} R^{(3)}_{\alpha\beta,\gamma\beta}\left[\sum_{k=0}^{n_u-1}(a^k_{\alpha\gamma} + b^k_{\alpha\gamma}) + \sum_{k=0}^{n_l-1}(c^k_{\alpha\gamma} + d^k_{\alpha\gamma})\right] \\
& - \frac{4}{3h^2} R^{(3)}_{\gamma\beta,\gamma\beta} - \frac{1}{2h} R^{(2)}_{\alpha\beta,\gamma\beta}\left[\sum_{k=0}^{n_u-1}(a^k_{\alpha\gamma} + b^k_{\alpha\gamma}) \right. \\
& \left. - \sum_{k=0}^{n_l-1}(c^k_{\alpha\gamma} + d^k_{\alpha\gamma})\right] - Q_{\alpha,\alpha} - \sum_{k=0}^{n_u-1}(a^k_{\alpha\gamma} + b^k_{\alpha\gamma}) Q^{(+k)}_{\alpha,\gamma} \\
& - \sum_{k=0}^{n_l-1}(c^k_{\alpha\gamma} + d^k_{\alpha\gamma}) Q^{(-k)}_{\alpha,\gamma} + \frac{1}{h} V^{(1)}_{\alpha,\gamma}\left[\sum_{k=0}^{n_u-1}(a^k_{\alpha\gamma} + b^k_{\alpha\gamma}) \right. \\
& \left. - \sum_{k=0}^{n_l-1}(c^k_{\alpha\gamma} + d^k_{\alpha\gamma})\right] + \frac{4}{h^2}\left\{V^{(2)}_{\alpha,\alpha} + ½\, V^{(2)}_{\alpha,\gamma}\left[\sum_{k=0}^{n_u-1}(a^k_{\alpha\gamma} + b^k_{\alpha\gamma}) \right.\right. \\
& \left.\left. + \sum_{k=0}^{n_l-1}(c^k_{\alpha\gamma} + d^k_{\alpha\gamma})\right]\right\} - p = 0
\end{aligned} \tag{13}$$

The boundary conditions are

$$\text{prescribed} \quad u^o_\alpha \quad \text{or} \quad N_{\alpha\beta}\nu_\beta \tag{14}$$

$$\text{prescribed} \quad \psi_\alpha \quad \text{or} \quad \hat{M}_{\alpha\beta}\nu_\beta \tag{15}$$

$$\text{prescribed} \quad \frac{\partial w}{\partial n} \quad \text{or} \quad \hat{R}^{(3)}_{\alpha\beta}\nu_\alpha\nu_\beta \tag{16}$$

prescribed w or

$$(\hat{Q}^\psi_\alpha + \hat{Q}^w_\alpha)\nu_\alpha + \frac{4}{3h^2}\left[\hat{R}^{(3)}_{\alpha\beta,\beta}\nu_\alpha - \frac{\partial}{\partial s}(\hat{R}^{(3)}_{\alpha\beta}\nu_\beta t_\alpha)\right] \tag{17}$$

where

$$\begin{aligned}
\hat{M}_{\alpha\beta} = & \sum_{k=0}^{n_u-1} a^k_{\gamma\alpha}(M^{+k}_{\gamma\beta} - z_k N^{+k}_{\gamma\beta}) + \sum_{k=0}^{n_l-1} c^k_{\gamma\alpha}(M^{-k}_{\gamma\beta} - \zeta_k N^{-k}_{\gamma\beta}) \\
& - \frac{2}{3h^2} R^{(3)}_{\gamma\beta}\left(\sum_{k=0}^{n_u-1} a^k_{\gamma\alpha} + \sum_{k=0}^{n_l-1} c^k_{\gamma\alpha}\right) \\
& - \frac{1}{2h} R^{(2)}_{\gamma\beta}\left(\sum_{k=0}^{n_u-1} a^k_{\gamma\alpha} - \sum_{k=0}^{n_l-1} c^k_{\gamma\alpha}\right)
\end{aligned} \tag{18}$$

$$\begin{aligned}
\frac{4}{3h^2} \hat{R}^{(3)}_{\alpha\beta} = & -\sum_{k=0}^{n_u-1} a^k_{\gamma\alpha}(M^{+k}_{\gamma\beta} - z_k N^{+k}_{\gamma\beta}) \\
& - \sum_{k=0}^{n_u-1} b^k_{\gamma\alpha}(M^{+k}_{\gamma\beta} - z_k N^{+k}_{\gamma\beta}) \\
& - \sum_{k=0}^{n_l-1} c^k_{\gamma\alpha}(M^{-k}_{\gamma\beta} - \zeta_k N^{-k}_{\gamma\beta}) \\
& - \sum_{k=0}^{n_l-1} d^k_{\gamma\alpha}(M^{-k}_{\gamma\beta} - \zeta_k N^{-k}_{\gamma\beta}) \\
& + \frac{2}{3h^2} R^{(3)}_{\gamma\beta}\left[\sum_{k=0}^{n_u-1}(a^k_{\gamma\alpha} + b^k_{\gamma\alpha}) + \sum_{k=0}^{n_l-1}(c^k_{\gamma\alpha} + d^k_{\gamma\alpha}) + 2\right] \\
& + \frac{1}{2h} R^{(2)}_{\gamma\beta}\left[\sum_{k=0}^{n_u-1}(a^k_{\gamma\alpha} + b^k_{\gamma\alpha}) - \sum_{k=0}^{n_l-1}(c^k_{\gamma\alpha} + d^k_{\gamma\alpha})\right]
\end{aligned} \tag{19}$$

$$\begin{aligned}
\hat{Q}^\psi_\alpha = & \sum_{k=0}^{n_u-1} a^k_{\gamma\alpha} Q^{+k}_\gamma + \sum_{k=0}^{n_l-1} c^k_{\gamma\alpha} Q^{-k}_\gamma - \frac{1}{h} V^{(1)}_\gamma\left(\sum_{k=0}^{n_u-1} a^k_{\gamma\alpha} - \sum_{k=0}^{n_l-1} c^k_{\gamma\alpha}\right) \\
& - \frac{2}{h^2} V^{(2)}_\gamma\left(\sum_{k=0}^{n_u-1} a^k_{\gamma\alpha} + \sum_{k=0}^{n_l-1} c^k_{\gamma\alpha}\right)
\end{aligned} \tag{20}$$

$$\begin{aligned}
\hat{Q}^w_\alpha = & Q_\alpha - \frac{4}{h^2} V^{(2)}_\alpha + \sum_{k=0}^{n_u-1} b^k_{\gamma\alpha} Q^{+k}_\gamma + \sum_{k=0}^{n_l-1} d^k_{\gamma\alpha} Q^{-k}_\gamma \\
& - \frac{1}{h} V^{(1)}_\gamma\left(\sum_{k=0}^{n_u-1} b^k_{\gamma\alpha} - \sum_{k=0}^{n_l-1} d^k_{\gamma\alpha}\right) - \frac{2}{h} V^{(2)}_\gamma\left(\sum_{k=0}^{n_u-1} b^k_{\gamma\alpha} - \sum_{k=0}^{n_l-1} d^k_{\gamma\alpha}\right)
\end{aligned} \tag{21}$$

Equilibrium equations can be reduced to simple forms when recast in terms of the resultants defined in Eqs. (18-21).

$$\begin{aligned}
\delta u^o_\alpha \;&:\; N_{\alpha\beta,\beta} = 0 \\
\delta\psi_\alpha \;&:\; -\hat{M}_{\alpha\beta,\beta} + \hat{Q}^\psi_\alpha = 0 \\
\delta w \;&:\; \frac{4}{3h^2}\hat{R}^{(3)}_{\gamma\beta,\gamma\beta} + (\hat{Q}^\psi_{\alpha,\alpha} + \hat{Q}^w_{\alpha,\alpha}) + p = 0
\end{aligned} \tag{22}$$

From the equilibrium equations and boundary conditions we see that the present theory is an eighth-order theory like

Reddy's[3] or Di Sciuva's.[5] At a force-prescribed boundary condition, the higher order twisting moment resultants contribute toward the effective resultant shear force as in Kirchhoff theory. At the edge, the in-plane force resultant is coupled with the bending moment resultants. This seems to be a characteristic of plate formulations for the general lamination configuration. This edge coupling did not occur in the symmetric lamination case.[10]

Numerical Results for Cylindrical Bending of Laminated Plates

To demonstrate and compare with other theories, the analytical solution for cylindrical bending is obtained. In this case, Eqs. (11–13) are reduced to one-dimensional form. In-plane stretching and bending resultants N_{11}, M_{11}^{+k}, M_{11}^{-k}, $R_{11}^{(2)}$, and $R_{11}^{(3)}$ are expressed in terms of the displacement variables $u_{1,1}^{o}$, $\psi_{1,1}$, and $w_{,11}$. Shear resultants Q_1, Q_1^{+k}, Q_1^{-k}, $V_1^{(1)}$, and $V_1^{(2)}$ are given in terms of rotational variables ψ_1 and $w_{,1}$.

The boundary conditions for simply supported ends are from conditions (14–17).

$$N_{11} = w = \hat{M}_{11} = \hat{R}_{11}^{(3)} = 0 \quad \text{at} \quad x_1 = 0, L \tag{23}$$

Constitutive relations and simply supported boundary condition (23) suggest the following form of displacements:

$$u(x) = \sum_{n=0}^{\infty} U_n \cos(n\alpha x)$$

$$w(x) = \sum_{n=0}^{\infty} W_n \sin(n\alpha x)$$

$$\psi_1(x) = \sum_{n=0}^{\infty} \Psi_n \cos(n\alpha x) \tag{24}$$

where $\alpha = \pi/L$. We expand the applied transverse load p in Fourier series

$$p = \sum_{n=1}^{\infty} p_n \sin(n\alpha x) \tag{25}$$

then Eqs. (24) satisfy boundary conditions (23). If we substitute Eqs. (24) and (25) into the equilibrium equations, we get

$$f_{11}U_n + f_{12}\Psi_n + f_{13}W_n = 0$$

$$f_{21}U_n + f_{22}\Psi_n + f_{23}W_n = 0$$

$$f_{31}U_n + f_{32}\Psi_n + f_{33}W_n = p_n \tag{26}$$

where detailed expressions of f_{ij} (where ij is 1, 2, and 3) are omitted because of space limitations.

To compare our results with Pagano's[11] exact solutions from Refs. 7 and 8, a single term loading $p = p_o \sin(\alpha x)$ will be considered. Two general laminations and a (0/90/0/90) asymmetric case are considered.

The material properties for the 0 deg layers are

$$E_1 = 25 \times 10^6 \text{ psi}, \qquad E_2 = 1 \times 10^6 \text{ psi}$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25, \qquad G_{12} = G_{13} = 0.5 \times 10^6 \text{ psi}$$

$$G_{23} = 0.2 \times 10^6 \text{ psi}$$

To facilitate comparison with other known theories,[1,4,7,8] the following nondimensional parameters are defined:

$$\bar{\sigma}_{11} = \frac{\sigma_{11}(L/2, z)}{p_o}$$

$$\bar{\sigma}_{13} = \frac{\sigma_{13}(0, z)}{p_o}$$

$$\bar{w} = \frac{100 E_1 h^3 w(L/2, 0)}{p_o L^4}$$

Table 2 Geometric and physical properties of arbitrary three- and five-layer laminates ($L/h = 4$)

Number of layers, N	Layer	Thickness ratio	Material
	1	0.25	3
3	2	0.40	1
	3	0.35	2
	1	0.10	1
	2	0.25	2
5	3	0.15	3
	4	0.20	1
	5	0.30	3

Table 3 Elastic properties of three different materials

	Material 1	Material 2	Material 3
Q_{11}/E_T[a]	1.002506	32.631	25.062657
Q_{55}/E_T	0.2	8.21	0.5

[a] E_T is the reference modulus.

Table 4 Center deflection $\bar{w}$ for arbitrary three- and five-layer laminates ($L/h = 4$)

	Elasticity	Present Theory	First-order zig-zag
$N=3$	2.341	2.200	1.992
$N=5$	2.456	2.249	1.261

In the (0/90/0/90) antisymmetric case, the first-order zig-zag theory[4] and Lo et al.[1] give poor estimates of the center deflections, whereas the present theory gives satisfactory values (see Table 1). The first-order zig-zag theory gives fairly good results in local and global behavior for the symmetric lamination case,[4] but it cannot be extended to the asymmetric case. The distributions through the thickness of in-plane displacement $\bar{u}_1$ and normal stress $\bar{\sigma}_{11}$ are shown in Figs. 2a and 2b for $L/h = 4$ and Figs. 3a and 3b for $L/h = 6$. From the $\bar{u}_1$ distributions in Figs. 2a and 3a, we can see that the first-order zig-zag solution deviates significantly at the bottom layer but the present theory shows good agreement with the exact solution. In-plane normal stress distributions are shown in Figs. 2b and 3b. Whereas the first-order zig-zag theory underestimates in-plane normal stress at the interface of the laminates, the present theory offers good in-plane stress distributions.

There are two ways to obtain transverse shear stresses, directly from the constitutive equation and by integrating the equilibrium equations of three-dimensional elasticity. Both methods give a continuous varying transverse shear stress distribution through the thickness, but the latter method gives better results than the former as in other plate theories. Figures 2c and 3c show the transverse shear stress distribution through the thickness. The method of integrating the equilibrium equations shows very accurate transverse shear results when compared to the corresponding exact solutions. The direct constitutive approach shows less accurate shear distribution. But even the constitutive approach predicts the maximum transverse shear stress quite accurately.

To further assess the range of applicability of the present theory, we consider problems involving three materials in three- and five-layer configurations. The geometric and physical properties are given in Table 2, and the material properties are given in Table 3. There are severe changes of the material properties in the transverse shear as well as in the in-plane axial direction. The elasticity solution for these problems has also been obtained in Ref. 8.

In Table 4, center deflections are compared. First-order zig-zag theory predicts the center deflections very poorly. The error in $\bar{w}$ for $N = 3$ is 15%, and for $N = 5$ is 49%. Figures 4a and 4b and 5a and 5b show the in-plane displacements and

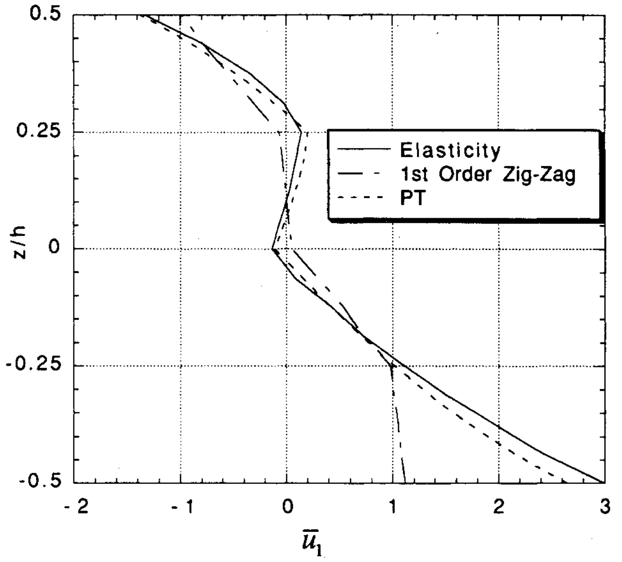

Fig. 2a In-plane displacement variation through the thickness of an antisymmetric four-layer cross-ply laminate for $L/h = 4$.

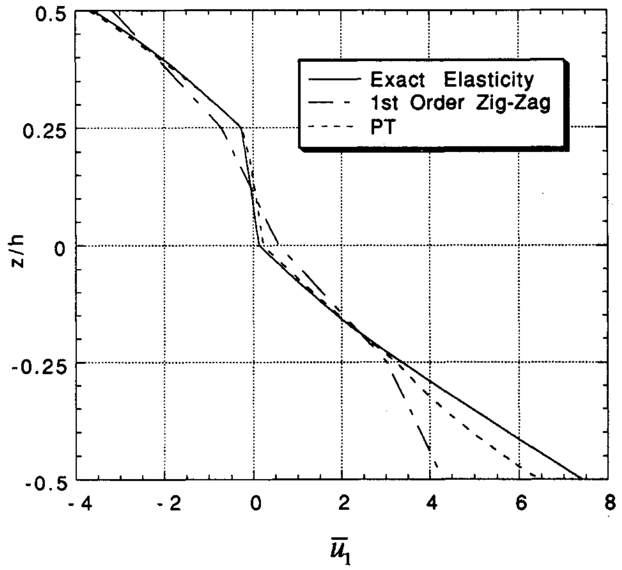

Fig. 3a In-plane displacement variation through the thickness of an antisymmetric four-layer cross-ply laminate for $L/h = 6$.

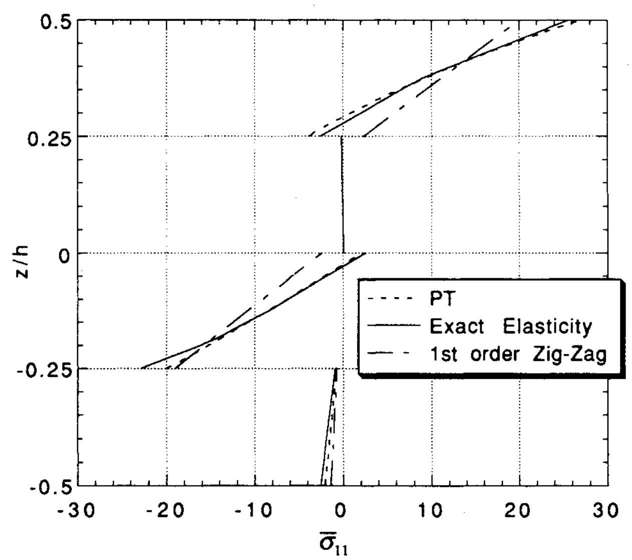

Fig. 2b In-plane normal stress variation through the thickness of an antisymmetric four-layer cross-ply laminate for $L/h = 4$.

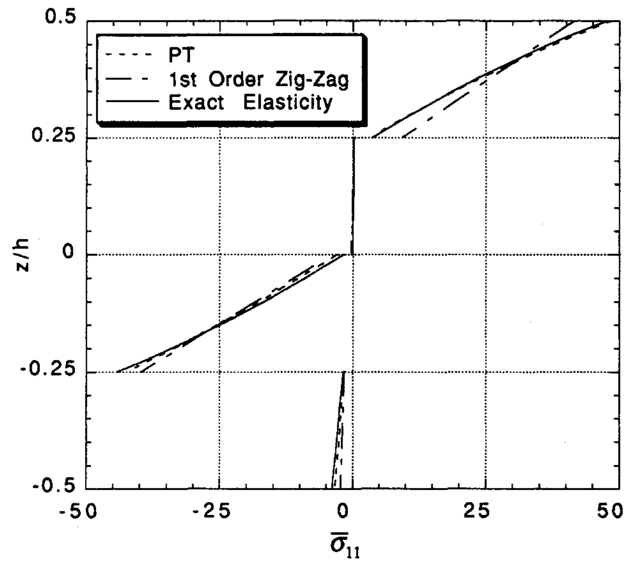

Fig. 3b In-plane normal stress variation through the thickness of an antisymmetric four-layer cross-ply laminate for $L/h = 6$.

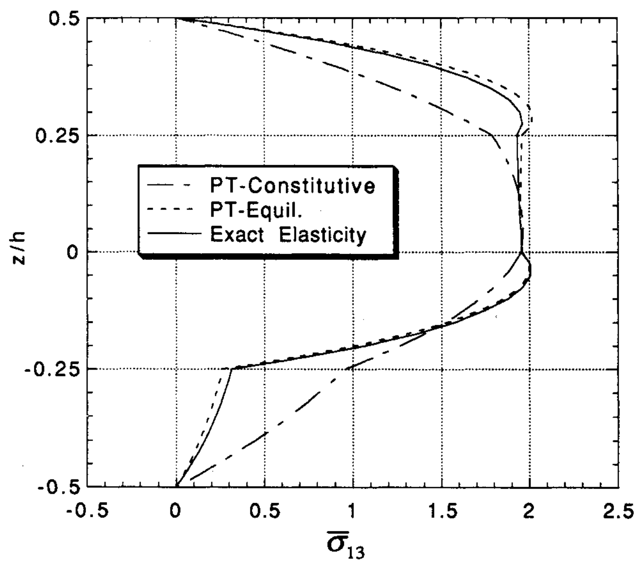

Fig. 2c Transverse shear stress variation through the thickness of an antisymmetric four-layer cross-ply laminate for $L/h = 4$.

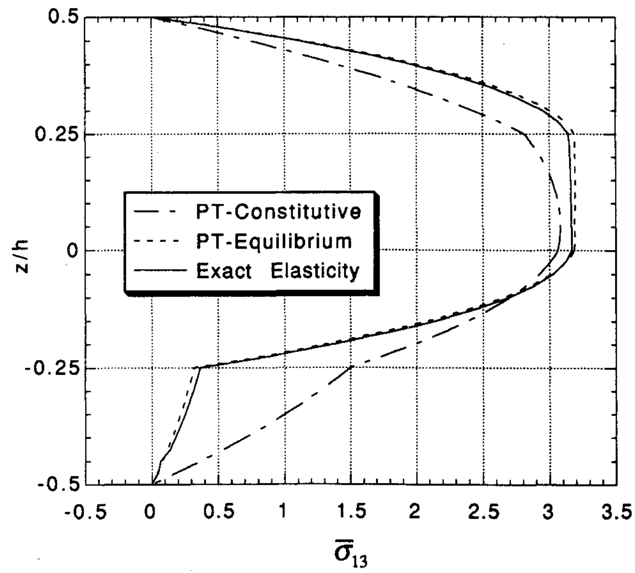

Fig. 3c Transverse shear stress variation through the thickness of an antisymmetric four-layer cross-ply laminate for $L/h = 6$.

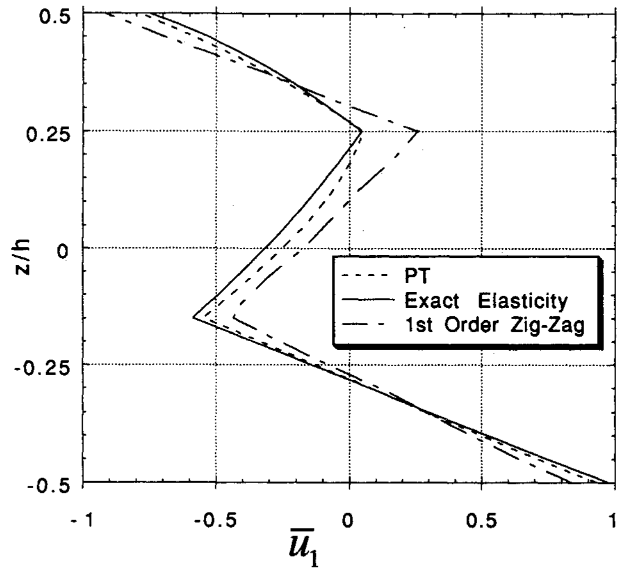

Fig. 4a In-plane displacement variation through the thickness of an arbitrary three-layer laminate for $L/h = 4$.

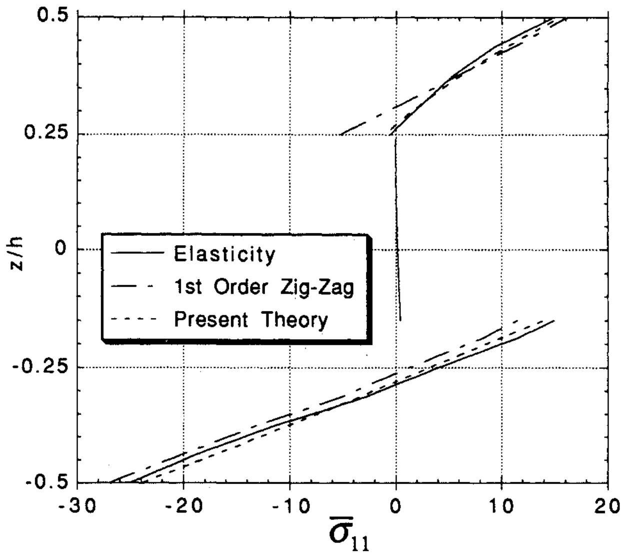

Fig. 4b In-plane normal stress variation through the thickness of an arbitrary three-layer laminate for $L/h = 4$.

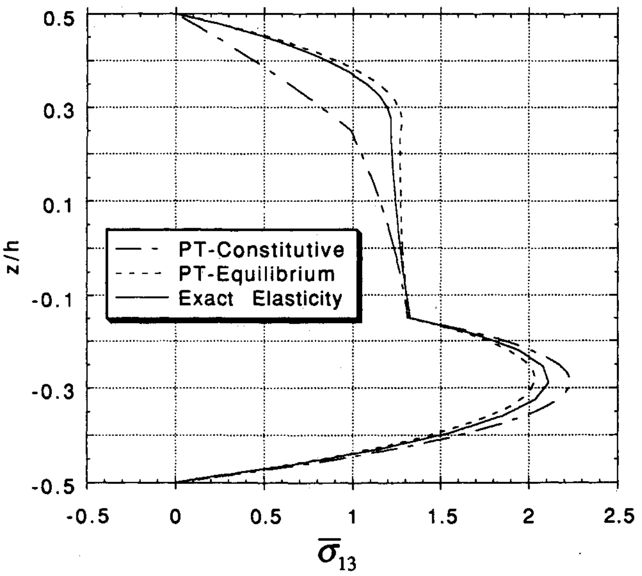

Fig. 4c Transverse shear stress variation through the thickness of an arbitrary three-layer laminate for $L/h = 4$.

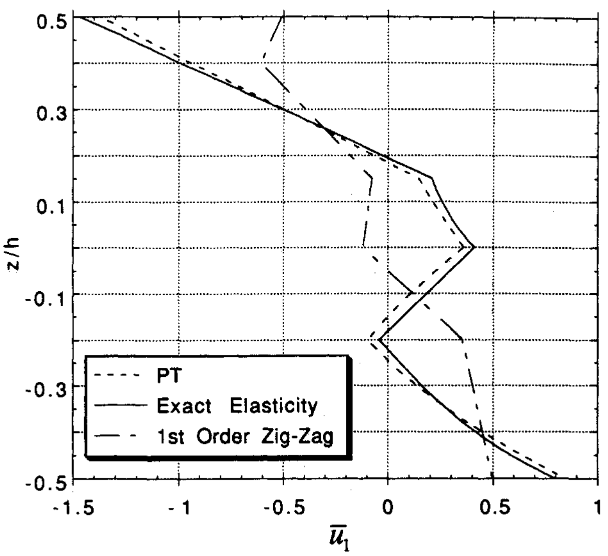

Fig. 5a In-plane displacement variation through the thickness of an arbitrary five-layer laminate for $L/h = 4$.

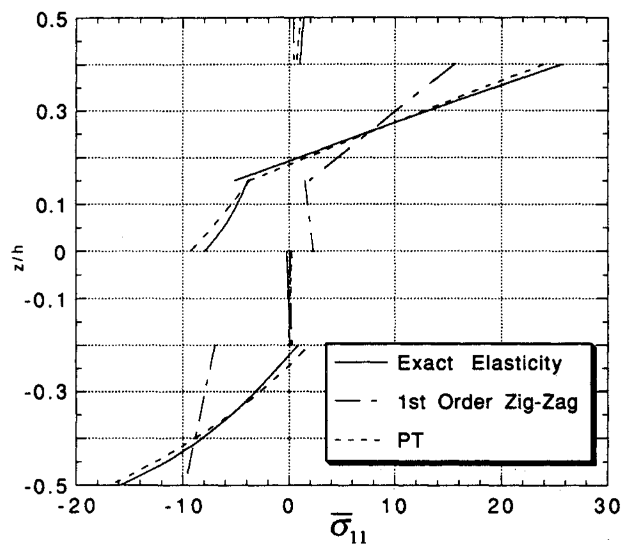

Fig. 5b In-plane normal stress variation through the thickness of an arbitrary five-layer laminate for $L/h = 4$.

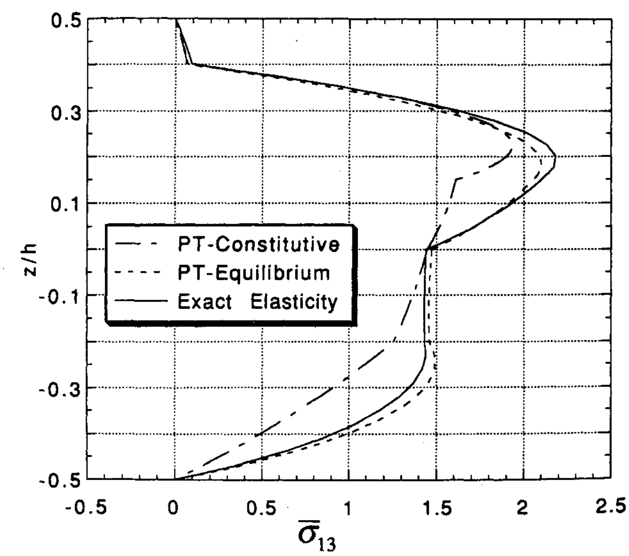

Fig. 5c Transverse shear stress variation through the thickness of an arbitrary five-layer laminate for $L/h = 4$.

stresses. First-order zig-zag theory fails to calculate the displacements and stresses accurately. Especially in the case of $N=5$, first-order zig-zag theory gives very poor results. However, the present theory predicts accurate stresses and displacements.

Figures 4c and 5c show the transverse shear stress distributions for $N=3$ and 5, respectively. Once more there is reasonable agreement between transverse shear stresses calculated by the present theory using direct constitutive relations and the exact solutions, and even better agreement is obtained using equilibrium equations.

Conclusion

The present theory has several advantages. As far as geometric boundary conditions are concerned, it introduces pointwise lateral deflection and rotations of normals to the midplane of the plate. Therefore, all of the dependent variables have clear geometrical meanings. The proposed in-plane displacement fields satisfy shear stress continuity conditions and shear free conditions on the upper and lower surfaces. Therefore, an artificial shear correction factor is not needed. The number of primary variables is the same as that of first-order shear deformation theories, and the number of dependent variables is independent of the number of layers. Therefore, it is efficient at analyzing problems with a large number of layers. With these advantages a displacement-based formulation can easily be implemented in a finite element. However, there are two drawbacks. The first is that, like other higher order plate theories, lengthy algebra is involved. The second is that, at force-prescribed boundaries, the theory must prescribe higher order resultant quantities that result from self-equilibrating stress distributions through the thickness.

The proposed theory has been verified by comparison with known exact cylindrical bending solutions. To apply the theory to more complicated problems with various geometries and boundary conditions, it is necessary to develop a finite element. This work is now in progress.

Appendix: Calculation of Terms in Equation (4)

The transverse shear stress continuity conditions are

$$\tau_{3\alpha}|_{z=z_m^-} = \tau_{3\alpha}|_{z=z_m^+}$$

where

$$m = 1, 2, \ldots, n_u - 1 \quad \text{for} \quad z>0$$

$$\tau_{3\alpha}|_{z=0^-} = \tau_{3\alpha}|_{z=0^+} \quad \text{for} \quad z=0$$

$$\tau_{3\alpha}|_{z=\zeta_m^-} = \tau_{3\alpha}|_{z=\zeta_m^+}$$

where

$$m = 1, 2, \ldots, n_l - 1 \quad \text{for} \quad z<0$$

These transverse shear stress continuity conditions at the interfaces can be expressed by the following matrix equation:

$$[A_{ij}]\left\{\begin{matrix} S_1^1 \\ S_2^1 \\ \vdots \\ S_1^{n_u-1} \\ S_2^{n_u-1} \\ T_1^0 \\ T_2^0 \\ \vdots \\ T_1^{n_l-1} \\ T_2^{n_l-1} \end{matrix}\right\}
= \left\{\begin{matrix}
-\Delta Q_{45}^{1}\left(1-\dfrac{z_1}{h}-\dfrac{2z_1^2}{h^2}\right) \\
-\Delta Q_{55}^{1}\left(1-\dfrac{z_1}{h}-\dfrac{2z_1^2}{h^2}\right) \\
\vdots \\
-\Delta Q_{45}^{n_u-1}\left(1-\dfrac{z_{n_u-1}}{h}-\dfrac{2z_{n_u-1}^2}{h^2}\right) \\
-\Delta Q_{55}^{n_u-1}\left(1-\dfrac{z_{n_u-1}}{h}-\dfrac{2z_{n_u-1}^2}{h^2}\right) \\
Q_{45}^1 \\
Q_{55}^1 \\
-\Delta Q_{45}^{-1}\left(-\dfrac{\zeta_1}{h}-\dfrac{2\zeta_1^2}{h^2}\right) \\
-\Delta Q_{55}^{-1}\left(-\dfrac{\zeta_1}{h}-\dfrac{2\zeta_1^2}{h^2}\right) \\
\vdots \\
-\Delta Q_{45}^{-n_l+1}\left(-\dfrac{\zeta_{n_l-1}}{h}-\dfrac{2\zeta_{n_l-1}^2}{h^2}\right) \\
-\Delta Q_{55}^{-n_l+1}\left(-\dfrac{\zeta_{n_l-1}}{h}-\dfrac{2\zeta_{n_l-1}^2}{h^2}\right)
\end{matrix}\right\}(\psi_1 + w_{,1})$$

$$+ \left\{\begin{matrix}
\Delta Q_{45}^{1}\left(-\dfrac{z_1}{h}+\dfrac{2z_1^2}{h^2}\right) \\
\Delta Q_{55}^{1}\left(-\dfrac{z_1}{h}+\dfrac{2z_1^2}{h^2}\right) \\
\vdots \\
\Delta Q_{45}^{n_u-1}\left(-\dfrac{z_{n_u-1}}{h}+\dfrac{2z_{n_u-1}^2}{h^2}\right) \\
\Delta Q_{55}^{n_u-1}\left(-\dfrac{z_{n_u-1}}{h}+\dfrac{2z_{n_u-1}^2}{h^2}\right) \\
-Q_{45}^1 \\
-Q_{55}^1 \\
-\Delta Q_{45}^{-1}\left(1+\dfrac{\zeta_1}{h}-\dfrac{2\zeta_1^2}{h^2}\right) \\
-\Delta Q_{55}^{-1}\left(1+\dfrac{\zeta_1}{h}-\dfrac{2\zeta_1^2}{h^2}\right) \\
\vdots \\
-\Delta Q_{45}^{-n_l+1}\left(1+\dfrac{\zeta_{n_l-1}}{h}-\dfrac{2\zeta_{n_l-1}^2}{h^2}\right) \\
-\Delta Q_{55}^{-n_l+1}\left(1+\dfrac{\zeta_{n_l-1}}{h}-\dfrac{2\zeta_{n_l-1}^2}{h^2}\right)
\end{matrix}\right\} w_{,1}$$

$$
+\begin{Bmatrix}
-\Delta Q_{44}^{1}\left(1-\frac{z_1}{h}-\frac{2z_1^2}{h^2}\right)\\
-\Delta Q_{45}^{1}\left(1-\frac{z_1}{h}-\frac{2z_1^2}{h^2}\right)\\
\vdots\\
-\Delta Q_{44}^{n_u-1}\left(1-\frac{z_{n_u-1}}{h}-\frac{2z_{n_u-1}^2}{h^2}\right)\\
-\Delta Q_{45}^{n_u-1}\left(1-\frac{z_{n_u-1}}{h}-\frac{2z_{n_u-1}^2}{h^2}\right)\\
Q_{44}^{1}\\
Q_{45}^{1}\\
-\Delta Q_{44}^{-1}\left(-\frac{\zeta_1}{h}-\frac{2\zeta_1^2}{h^2}\right)\\
-\Delta Q_{45}^{-1}\left(-\frac{\zeta_1}{h}-\frac{2\zeta_1^2}{h^2}\right)\\
\vdots\\
-\Delta Q_{44}^{-n_l+1}\left(-\frac{\zeta_{n_l-1}}{h}-\frac{2\zeta_{n_l-1}^2}{h^2}\right)\\
-\Delta Q_{45}^{-n_l+1}\left(-\frac{\zeta_{n_l-1}}{h}-\frac{2\zeta_{n_l-1}^2}{h^2}\right)
\end{Bmatrix}(\psi_2+w_{,2})
$$

$$
+\begin{Bmatrix}
\Delta Q_{44}^{1}\left(-\frac{z_1}{h}+\frac{2z_1^2}{h^2}\right)\\
\Delta Q_{45}^{1}\left(-\frac{z_1}{h}+\frac{2z_1^2}{h^2}\right)\\
\vdots\\
\Delta Q_{44}^{n_u-1}\left(-\frac{z_{n_u-1}}{h}+\frac{2z_{n-1}^2}{h^2}\right)\\
\Delta Q_{45}^{n_u-1}\left(-\frac{z_{n_u-1}}{h}+\frac{2z_{n_u-1}^2}{h^2}\right)\\
-Q_{44}^{1}\\
-Q_{45}^{1}\\
-\Delta Q_{44}^{-1}\left(1+\frac{\zeta_1}{h}-\frac{2\zeta_1^2}{h^2}\right)\\
-\Delta Q_{45}^{-1}\left(1+\frac{\zeta_1}{h}-\frac{2\zeta_1^2}{h^2}\right)\\
\vdots\\
-\Delta Q_{44}^{-n_l+1}\left(1+\frac{\zeta_{n_l-1}}{h}-\frac{2\zeta_{n_l-1}^2}{h^2}\right)\\
-\Delta Q_{45}^{-n_l+1}\left(1+\frac{\zeta_{n_l-1}}{h}-\frac{2\zeta_{n_l-1}^2}{h^2}\right)
\end{Bmatrix}w_{,2}
$$

where

$$\Delta Q_{ij}^{m} = Q_{ij}^{(m+1)} - Q_{ij}^{(m)}$$

$[A_{ij}]$ is a $2(n_l + n_u - 1) \times 2(n_l + n_u - 1)$ square matrix. The detailed lengthy expression of $[A_{ij}]$ is omitted for the limited space.

These equations can be written as following form

$$[A]\{S,T\}^T = \{B_1\}(\psi_1 + w_{,1}) + \{B_2\}w_{,1} + \{B_3\}(\psi_2 + w_{,2}) + \{B_4\}w_{,2}$$

$$\{S,T\}^T = [A]^{-1}\{B_1\}(\psi_1 + w_{,1}) + [A]^{-1}\{B_2\}w_{,1} + [A]^{-1}\{B_3\}(\psi_2 + w_{,2}) + [A]^{-1}\{B_4\}w_{,2}$$

Therefore

$$S_\alpha^k = a_{\alpha\gamma}^k(w_{,\gamma} + \psi_\gamma) + b_{\alpha\gamma}^k w_{,\gamma}$$

$$T_\alpha^k = c_{\alpha\gamma}^k(w_{,\gamma} + \psi_\gamma) + d_{\alpha\gamma}^k w_{,\gamma}$$

Acknowledgment

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